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Pressure of hot QCD at large N_f

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Abstract: We compute the pressure and entropy of hot QCD in the limit of large number of fermions, $N_\mathrm{f} \gg N_\mathrm{c} \sim 1$, to next to leading order in N_f. At this order the calculation can be done exactly, up to ambiguities due to the presence of a Landau pole in the theory; the ambiguities are $O(T^8/\Lambda^4_{\mathrm{Landau}})$ and remain negligible long after the perturbative series (in $g^2 N_\mathrm{f}$) has broken down. Our results can be used to test several proposed resummation schemes for the pressure of full QCD.

Keywords: 1/N Expansion, Thermal Field Theory, QCD.

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1. Introduction

At temperatures large compared to the intrinsic scale of QCD, $T \gg \Lambda_{\rm QCD}$, the running QCD coupling constant is small, and perturbation theory should be a useful tool for studying thermodynamic and dynamic properties of the theory. This is true in principle, at least.

In practice, there are obstacles to the use of perturbation theory. The first is that hot QCD has an intrinsically nonperturbative scale, at wave numbers $k \sim g^2 T$, or length scales $l \sim 1/g^2 T$. (The running coupling g^2 should be evaluated at a scale of order T.) The presence of nonperturbative physics at this scale has nothing to do with asymptotic freedom or the existence of the scale $\Lambda_{\rm QCD}$; even at temperatures many orders of magnitude above $\Lambda_{\rm QCD}$, nonperturbative physics sets in at a scale only a little lower than the thermal scale, $g^2 T \sim T/\log(T/\Lambda_{\rm QCD})$. Some observables are not very sensitive to this infrared scale; but any observable will be sensitive to this scale at some order in the coupling expansion. For instance, the Debye screening length carries nonperturbative corrections which are suppressed by only one power of g [1, 2]; photon emissivity is believed to receive nonperturbative corrections suppressed by g^2 [3]; and the baryon number violation rate arises entirely from nonperturbative physics,and is sensitive at leading order [4].

The measurables which show thermal effects with the least sensitivity to the nonperturbative scale are the standard thermodynamic parameters, such as pressure, entropy density, and energy density. For these quantities, perturbation theory can determine all coefficients up to $O(g^5)$, and that calculation has been carried out [5]–[7]. (It is also possible to compute the $O(g^6 \ln(g))$ term, and this has recently been completed [8].) With the coefficients of the pressure up to $O(g^5)$ in hand, one can diagnose the quality of the perturbative expansion, by seeing how quickly successive partial sums converge. Here we meet the other problem of perturbation theory at finite temperature; even when the perturbative series exists to some order, it shows very poor convergence. This problem is distinct from the presence of nonperturbative magnetic physics; for instance it also appears in hot QED, which is free of strongly coupled IR behavior [9, 5].

Recently this problem has received renewed attention. Three groups have proposed reorganizations or partial resummations of the thermal perturbative expansion, which are hoped to improve the convergence, and therefore the utility, of perturbation theory. Andersen, Braaten, and Strickland, following an earlier proposal by Karsch et al. in scalar field theories [10], have proposed to do so by adding the hard thermal loop (HTL) lagrangian to the action at tree level and subtracting it again with a counterterm treated as formally higher order, a technique they call hard thermal loop perturbation theory [11]–[13]. Blaizot, Iancu, and Rebhan have proposed a technique based on 2PI Φ derivable methods, together with a set of approximations to render the method tractable, also based on hard thermal loops [14]–[16]. A similar proposal was recently made by Peshier [17].

All three groups have produced predictions for the QCD pressure, both for pure glue and for QCD with fermions. Unfortunately, it is difficult to judge whether the techniques are successful, since all we know independently about the QCD pressure are the perturbative series, which is only useful at uninterestingly weak coupling, and lattice results from the Bielefeld group [18], which are only at such large coupling that it is not clear whether the resummation techniques should still work. The resummation techniques have also been applied to scalar field theory, but in that case exact results are only known for $\mathrm{O}(N)$ scalar theory at large N, which is an almost trivial theory.

For this reason we think it advantageous to have accurate results for the pressure, in a theory as close to QCD as possible. In this paper we present a calculation of the thermal pressure in QCD with a large number of fermions, $N_{\mathrm{f}} \gg N_{\mathrm{c}} \sim 1$, at next to leading order (NLO) in a large N_{f} expansion and for general $g^2 N_{\mathrm{f}}$, which is treated as $O(1)$. This theory contains a great deal of the physics of QCD; for instance, the gauge fields are screened by the plasma with the same complicated frequency and momentum dependence as in full QCD. The perturbative series for the pressure shows poor behavior, as in full QCD. Of course, some of the interesting physics of full QCD is absent at NLO in this expansion. This is unfortunate; but on the other hand it is what allows us to solve this theory. For previous exact results in 6 dimensional scalar theory, see [19] (who encountered some similar issues, regarding Landau singularities, to those present here).

We should emphasize now that, since all of the physics of large N_{f} QCD is also present in full QCD, any approximation which shows poor performance in large N_{f} QCD cannot be expected to be valid for full QCD either. The reverse need not be the case; an approximation which works at large N_{f} may stumble over the physics which is left out in the large N_{f} expansion. Therefore, an approximation's performance against our results should be considered an optimistic estimate of its performance in full QCD.

In the next section, we will explain what we mean by large N_{f} QCD. In section 3, we describe how the calculation of the QCD pressure is carried out in this theory. Our results and conclusions are presented in section 4, but we very briefly summarize them here. At small effective coupling $g^2 N_{\mathrm{f}}$, the pressure is close to the free theory value, and perturbation theory works well. As the effective coupling is increased, P/T^4 falls at first, but eventually rises, becoming larger than the free theory value well before the theory

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encounters its Landau pole. The entropy shows the same behavior. It is impossible to obtain such behavior in an ideal quasiparticle approximation, such as the model of Peshier, Kämpfer, Pavlenko, and Soff [20].

2. Large $N_{\rm f}$ QCD

By large $N_{\rm f}$ QCD, we mean QCD with the number of colors $N_{\rm c}$ taken as fixed and $O(1)$, but with the number of fermionic species $N_{\rm f}$ taken large and the gauge coupling g^2 taken small, so that the combination $g^2 N_{\rm f}$, which serves as an effective coupling strength, remains fixed and $O(1)$. Therefore, in $N_{\rm f}$ counting, a factor of g^2 is treated as $O(1/N_{\rm f})$. This procedure is appropriate because it is the effective coupling which determines the convergence of perturbation theory.

In the following it will be useful to define $g^2_{\rm eff}$ as

$$g^2_{\rm eff} \equiv \frac{g^2 N_{\rm f} C_{\rm F} d_{\rm F}}{d_{\rm A}} = \begin{cases} \dfrac{g^2 N_{\rm f}}{2}\,, \ \mathrm{QCD}\,, \\ \\ g^2 N_{\rm f}\,, \ \mathrm{QED}\,. \end{cases} \tag{2.1}$$

Here $C_{\rm F}$ is the quadratic Casimir of the representation containing the fermions and $d_{\rm F}$ and $d_{\rm A}$ are the dimensions of the fermionic and adjoint representations. In $\mathrm{SU}(N_{\rm c})$ gauge theory but with adjoint fermions, $g^2_{\rm eff} = g^2 N_{\rm f} N_{\rm c}$. Other than overall factors involving $N_{\rm c}$, all the nontrivial coupling dependence of the pressure and entropy which we will find will depend on $g^2_{\rm eff}$. Note that $g^2_{\rm eff}$ is renormalization point dependent; at leading order in $N_{\rm f}$ its renormalization point dependence is,

$$\beta(g^2_{\rm eff}) \equiv \frac{\mu\, dg^2_{\rm eff}}{d\mu} = \frac{(g^2_{\rm eff})^2}{6\pi^2}\,. \tag{2.2}$$

There are no corrections involving higher powers of $g^2_{\rm eff}$; all corrections to this relation are suppressed by at least one power of $1/N_{\rm f}$. Therefore, one can solve completely for the scale dependence of $g^2_{\rm eff}$, at leading order in $1/N_{\rm f}$;

$$\frac{1}{g^2_{\rm eff}(\mu)} = \frac{1}{g^2_{\rm eff}(\mu')} + \frac{\ln(\mu'/\mu)}{6\pi^2}\,. \tag{2.3}$$

Clearly the theory has a Landau pole at $\Lambda_{\rm Landau} \sim \mu e^{6\pi^2/g^2_{\rm eff}}$. We will define $\Lambda_{\rm Landau}$ so that the gauge field propagator diverges at $Q^2 = \Lambda^2_{\rm Landau}$; under this definition, $\Lambda_{\rm Landau} = \mu e^{5/6} e^{6\pi^2/g^2_{\rm eff}}$. There is no hope that some strong coupling dynamics somehow generate a strongly coupled UV fixed point, because eq. (2.3) is exact at leading order in $1/N_{\rm f}$. Therefore the theory technically does not exist.

This problem is fatal if $\Lambda_{\rm Landau} \sim T$; but for $\Lambda_{\rm Landau} \gg T$ we argue that it is not. The Landau pole means that the theory as such becomes ill defined; but it is possible to "rescue" the theory by introducing either additional heavy degrees of freedom or high dimension operators, if they arise at a scale $\leq \Lambda_{\rm Landau}$. For instance, if the operator $F_{\mu\nu} D^2 D^2 F^{\mu\nu}/\Lambda^4_{\rm Landau}$ is added to the lagrangian with coefficient greater than 1, the gauge

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propagator becomes nonsingular at all energies, which is sufficient for our NLO calculation to proceed. This term ruins the renormalizability of the theory, but in a way which will not appear in the NLO calculation of the pressure. What the Landau pole does mean is, that the exact definition of the theory is ambiguous, and quantities like the entropy or the pressure are only completely well defined after the physics which resolves the Landau singularity has been specified. The size of the ambiguity is, however, suppressed by a positive power of $(T/\Lambda_{\rm Landau})$, which for the pressure at NLO turns out to be $(T/\Lambda_{\rm Landau})^4$. So provided that $\Lambda_{\rm Landau}$ is kept suitably larger than T, the relative size of the ambiguity is tiny and can be ignored.[1] As $g^2_{\rm eff}$ is increased, this new ambiguity remains tiny long after the convergence of perturbation theory for the pressure has broken down. Therefore, we will forge ahead and compute the pressure, without specifying the UV completion; but we will only work with values of $g^2_{\rm eff}(\mu)$ for which $(\Lambda_{\rm Landau}/T) > 35$, which proves abundantly sufficient to keep the relative ambiguity in the NLO pressure below 1%.

The other feature of large $N_{\rm f}$ gauge theory which we should emphasize is, that the nonabelian nature of the theory is not very important. This is because the gauge field self-interactions carry powers of g without introducing powers of $N_{\rm f}$. In particular the gauge self-interactions will have no relevance in the calculation we perform here, and first appear, in the pressure and entropy, at NNLO (next to next to leading order) in $1/N_{\rm f}$. Therefore our results will apply both to QED and QCD.

3. Computation of the pressure

In a relativistic field theory with vanishing chemical potentials, the pressure and free energy density are opposites, $P = -F$; so we compute the pressure by computing the free energy, which equals the trace of the sum of all 1PI vacuum bubble graphs. This is to be done at finite temperature, which means in euclidean space with time extent $\beta = 1/T$, periodic boundary conditions for bosons, and antiperiodic boundary conditions for fermions. To get the *thermal* pressure, we subtract off the result at zero temperature; the difference will be finite, though renormalization is still necessary because the result depends on the coupling $g^2_{\rm eff}$ and this coupling runs.

At leading order, $O(N_{\rm f})$, there is only one diagram, a bare fermionic loop, shown in figure 1(A). Its contribution to the pressure is[2]

$$P_{\rm LO} = N_{\rm f}\ {\rm Tr}\left(\sum\!\!\!\!\!\!\int_Q^{\rm F} \ln \not{Q} - \int_Q \ln \not{Q} \right) . \tag{3.1}$$

Here the trace is over group and Dirac indices, and the fermionic sum-integral symbol means

$$\sum\!\!\!\!\!\!\int_Q^{\rm F} \equiv T \sum_{q^0=(2n+1)\pi T}^{n\in\mathcal{Z}} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \, ; \qquad \int_Q \equiv \int \frac{d^4Q}{(2\pi)^4} \, . \tag{3.2}$$

[1]Similar considerations, in the context of scalar field theory, appear in Drummond et al. [21].

[2]We use a $[++++]$ euclidean and $[-+++]$ Minkowski metric. Four-vectors are written with upper case letters, their space and time components in lower case, so if Q is a 4-vector, then q^0 is its time component, $\boldsymbol{q}$ is the spatial vector and $q \equiv |\boldsymbol{q}|$ its magnitude. The Minkowski continued frequency is typically denoted with a different letter, usually ω.

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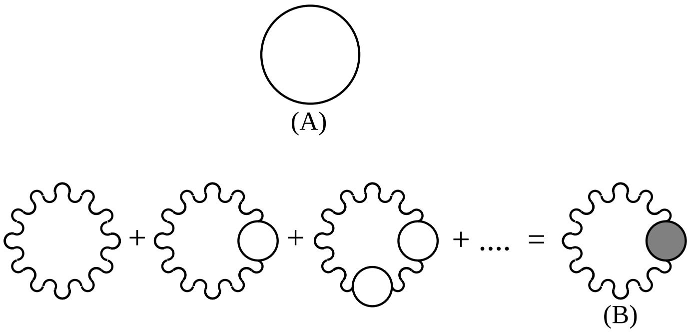

Figure 1: Leading order, $O(N_\mathrm{f})$ diagram, (A), which is a free fermion bubble, and NLO, $O(N_\mathrm{f}^0)$ diagrams, (B), which are a gauge boson loop with any number of fermion bubble self-energy insertions. (Coupling counterterm insertions, though not shown, should also be resummed.) They can be resummed by the usual Schwinger-Dyson method, to give a loop with a resummed gauge field propagator.

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We have written this in 3+1 dimensions, but dimensional regularization is implied. At leading order, since the $T = 0$ value is subtracted off, this is irrelevant, the result is finite and equals

$$P_\mathrm{LO} = 4N_\mathrm{f}\, d_\mathrm{F}\, \frac{7}{8}\, \frac{\pi^2 T^4}{90}\,. \tag{3.3}$$

This is the usual free theory value; $4N_\mathrm{f} d_\mathrm{F}$ counts the number of degrees of freedom [the 4 counts the two spin states each for particle and anti-particle; d_F is the number of colors, $d_\mathrm{F} = N_\mathrm{c}$ for fundamental representation fermions in $\mathrm{SU}(N_\mathrm{c})$].

At NLO, there are an infinite number of diagrams, but the structure is extremely simple; all contributions look like a gauge boson propagator with an arbitrary number of fermion loops, and counterterm insertions, inserted. No other graphs contribute at $O(N_\mathrm{f}^0)$. This is because the only way to add a loop to a graph without changing the power of N_f, is to add a fermion self-energy loop (or the required counterterm) to a gauge boson propagator; any other way to add a loop introduces a g^2 without another factor of N_f, and $g^2 \sim 1/N_\mathrm{f}$. This is the reason that large N_f gauge theory is so simple; the large N_f limit is a much stronger organizing principle than large N_c.

This set of diagrams can of course be resummed by the standard Schwinger-Dyson trick. Therefore the NLO contribution to the pressure is given by

$$P_\mathrm{NLO} = -\frac{1}{2}\,\mathrm{Tr}\left(\sum\!\!\!\!\!\!\!\int_Q^{\mathrm{B}} \ln\left[[G_0^{-1}]^{\mu\nu}(Q) + \Pi^{\mu\nu}_\mathrm{th}(Q)\right] - \int_Q \ln\left[[G_0^{-1}]^{\mu\nu}(Q) + \Pi^{\mu\nu}_\mathrm{vac}(Q)\right]\right), \tag{3.4}$$

where the trace runs over group and Lorentz indices. The group trace is trivial, it gives d_A [which is $N_\mathrm{c}^2 - 1$ for $\mathrm{SU}(N_\mathrm{c})$ and 1 for QED]. The bosonic sum-integral symbol means

$$\sum\!\!\!\!\!\!\!\int_Q^{\mathrm{B}} \equiv T \sum_{Q_0 = 2n\pi T}^{n\in\mathcal{Z}} \int \frac{d^3q}{(2\pi)^3}\,, \tag{3.5}$$

$\Pi^{\mu\nu}_\mathrm{th}$ is the fermionic plus counterterm contribution to the gauge boson self-energy at finite temperature, $\Pi^{\mu\nu}_\mathrm{vac}$ is the vacuum value, and G_0^{-1} is the free inverse propagator. Our sign convention for Π is chosen so that our Π agrees with the usual Minkowski, $[+---]$ Π. The same expression for the pressure can be derived from the Luttinger-Ward relation [22], with a comparable amount of work.

Both vacuum and thermal self-energies are gauge invariant, since their evaluation doesn't involve gauge field propagators. They are also exactly transverse, $Q_\mu \Pi^{\mu\nu}(Q) = 0$, and together with rotational invariance this ensures that $\Pi^{\mu\nu}_{\rm th} - \Pi^{\mu\nu}_{\rm vac}$ can be expressed in terms of two scalar functions [23],

$$\Pi^{\mu\nu}_{\rm th}(Q) - \Pi^{\mu\nu}_{\rm vac}(Q) = P^{\mu\nu}\Pi_{\rm T}(Q) + Q^{\mu\nu}\Pi_{\rm L}(Q)\,. \tag{3.6}$$

The exact form of $P^{\mu\nu}$ and $Q^{\mu\nu}$ are not important for our problem, all that matters to us is that they are properly normalized projection operators, $P^\mu_\mu = 2$, $Q^\mu_\mu = 1$, and $P^{\mu\nu}(Q) + Q^{\mu\nu}(Q) + Q^\mu Q^\nu/Q^2 = \eta^{\mu\nu}$ (which is $\delta^{\mu\nu}$, since right now we are in euclidean space). For the vacuum self-energy, the transverse and longitudinal components are the same, and their value is well known;

$$\Pi^{\mu\nu}_{\rm vac} = (P^{\mu\nu} + Q^{\mu\nu})\,\Pi_{\rm vac}\,, \qquad \Pi_{\rm vac} = -\frac{g^2_{\rm eff}\,Q^2}{12\pi^2}\left(\ln\frac{Q^2}{\overline{\mu}^2} - \frac{5}{3}\right). \tag{3.7}$$

Here $\overline{\mu}$ is the $\overline{\rm MS}$ renormalization point.

Some gauge fixing prescription is necessary to evaluate eq. (3.4), though the result is gauge invariant since it represents a complete treatment to some order in a systematic expansion (this can be explicitly checked). If we choose a gauge where G_0^{-1} factorizes in the same way as Π, the Lorentz trace and logarithm will commute. For instance, in Feynman gauge, $[G_0^{-1}]^{\mu\nu}(Q) = \eta^{\mu\nu}Q^2$, and (dropping the overall group factor)

$${\rm Tr}\ln\left[[G_0^{-1}]^{\mu\nu}(Q) + \Pi^{\mu\nu}_{\rm th}(Q)\right] = 2\ln(Q^2{+}\Pi_{\rm T}{+}\Pi_{\rm vac}){+}\ln(Q^2{+}\Pi_{\rm L}{+}\Pi_{\rm vac}){+}\ln(Q^2){-}2\ln(Q^2)\,. \tag{3.8}$$

The contribution containing $\Pi_{\rm T}$ comes from $P^{\mu\nu}$; the contribution containing $\Pi_{\rm L}$ comes from $Q^{\mu\nu}$; the positive $\ln(Q^2)$ contribution comes from $Q^\mu Q^\nu/Q^2$, and the $-2\ln(Q^2)$ contribution comes from the ghosts, which have no self-energy at this order in $N_{\rm f}$. In a general covariant gauge the $Q^\mu Q^\nu/Q^2$ term would be shifted by $\ln(\xi)$ with ξ the gauge fixing parameter; but this contributes equally to the vacuum and thermal contributions and so drops out.[3] The next task is to evaluate the thermal self-energy. This has been done by Weldon [23] (in Minkowski space, but the continuation is elementary). His result is listed in the appendix; unfortunately it involves an integration which to our knowledge cannot be done in closed form.

Because the self-energies $\Pi_{\rm L}$, $\Pi_{\rm T}$ are only expressed as single integrals and not as closed form, analytic expressions, a this point we must either make approximations, or turn to semi-numerical methods. We want exact results; otherwise there is no point in our looking at this theory. So the calculation will have to become more numerical. The difference between a sum-integral and an integral is an inconvenient quantity for numerical analysis,

[3]This is because $T\sum_{q^0} C = \int (dq^0/2\pi)C$ for any C which is q^0 independent. The easiest way to see this is directly in the time domain, though it also follows easily on rotating to Minkowski frequency. The result for $P_{\rm NLO}$ in Coulomb gauge differs by a factor of $\ln(q^2)$ from the covariant result, but this also cancels between vacuum and thermal for the same reason.

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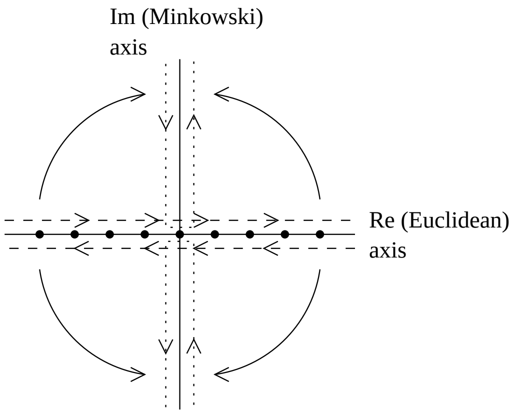

Figure 2: Points where summation occurs (dots on euclidean axis) become poles whose residues must be summed. The dashed contour sums these poles, and can be continued (arrows) to the dotted contour, which evaluates the discontinuity across the imaginary (Minkowski) axis.

especially when issues of UV regulation have not all been addressed. At large values of Q^2, the argument in the integral in eq. (3.4) is smooth, and the difference between a sum-integral and an integral is exponentially suppressed, by $O(\exp(-Q/T))$. So it is only the moderate q^2 region where we need to handle sum-integration. In this region, this can be done by re-expressing the frequency sum as a contour integration and continuing to Minkowski frequency. Namely, we note that the function $\cot(q^0/2T)$ has poles at $q^0 = 2n\pi T$, of residue $4\pi i T$. So the Matsubara sum can be replaced by a contour integral,

$$\sum_{q^0=2\pi nT}^{n\in\mathcal{Z}} \ln(Q^2+\Pi(Q)) = i\int_C \frac{dq^0}{4\pi}\cot\left(\frac{q^0}{2T}\right)\ln(Q^2+\Pi)\,, \tag{3.9}$$

where the contour runs just above the real axis from $-\infty$ to $+\infty$, and then just below the axis back to $-\infty$. This is illustrated in figure 2, which also shows that this contour can be deformed to pinch the imaginary (Minkowski) axis. The result from upper and lower half planes are equal, so the contour integral becomes

$$\sum_{q^0=2\pi nT}^{n\in\mathcal{Z}} \ln(Q^2+\Pi(Q)) = \int_0^\infty \frac{d\omega}{\pi}\coth\left(\frac{\omega}{2T}\right)\,\mathrm{Im}\,\ln\left(q^2-\omega^2+\Pi(q,i\omega+0)\right). \tag{3.10}$$

Note that $\coth(\omega/2T) = 1 + 2/(e^{\omega/T}-1) = 2(n_\mathrm{b}(\omega)+1/2)$. The $T\to 0$ limit is trivial; just leave out the Bose distribution and use the vacuum self-energy. Therefore, eq. (3.4) becomes

$$\begin{aligned} P_\mathrm{NLO} = d_\mathrm{A}\int\frac{d^3q}{(2\pi)^3}\int_0^\infty\frac{d\omega}{\pi}\Bigg[2\Bigg(\left[n_\mathrm{b}+\frac{1}{2}\right]&\mathrm{Im}\,\ln\left(q^2-\omega^2+\Pi_\mathrm{T}+\Pi_\mathrm{vac}\right) - \\ &-\frac{1}{2}\mathrm{Im}\,\ln\left(q^2-\omega^2+\Pi_\mathrm{vac}\right)\Bigg) + \\ &+\Bigg(\left[n_\mathrm{b}+\frac{1}{2}\right]\mathrm{Im}\,\ln\frac{q^2-\omega^2+\Pi_\mathrm{L}+\Pi_\mathrm{vac}}{q^2-\omega^2} - \\ &-\frac{1}{2}\mathrm{Im}\,\ln\frac{q^2-\omega^2+\Pi_\mathrm{vac}}{q^2-\omega^2}\Bigg)\Bigg]. \end{aligned} \tag{3.11}$$

Here Π is always understood to be evaluated at $(q, i\omega+0)$. Note that $\mathrm{Im}\,\Pi_{\rm vac} = (\omega^2-q^2)\times (g_{\rm eff}^2/12\pi)\theta(\omega^2-q^2)$ is nonzero for timelike momenta; the imaginary parts of the logarithms are always at least $O(g_{\rm eff}^2)$ away from π. There are no infrared problems in evaluating the above expression; we discuss ultraviolet issues next.

The thermal and vacuum contributions fail to cancel for two reasons. First, there is the factor of $n_{\rm b}(\omega)$ associated with the thermal contribution, which can be traced to the difference between $\sum_{q^0}$ and $\int dq^0$. There are no ultraviolet problems associated with this piece, since at large ω it is exponentially suppressed by $O(\exp(-\omega/T))$, while at large q but moderate ω, the thermal self-energy has an exponentially small imaginary part, $\mathrm{Im}\Pi \sim \exp(-(q-\omega)/2T)$. Therefore this piece is exponentially ultraviolet safe.

Next, there are $\Pi_{\rm T}$ and $\Pi_{\rm L}$. They are only power suppressed at large Q, and can potentially cause more trouble. To understand this region it is best to go back to the euclidean expression,[4] and expand in small T^2/Q^2. In this small T^2/Q^2 limit, the logarithms can be expanded,

$$2\ln\frac{Q^2+\Pi_{\rm vac}+\Pi_{\rm T}}{Q^2+\Pi_{\rm vac}} + \ln\frac{Q^2+\Pi_{\rm vac}+\Pi_{\rm L}}{Q^2+\Pi_{\rm vac}} = \frac{2\Pi_{\rm T}+\Pi_{\rm L}}{Q^2+\Pi_{\rm vac}} - \frac{2(\Pi_{\rm T})^2+\Pi_{\rm L}^2}{2(Q^2+\Pi_{\rm vac})^2} + \cdots\,. \tag{3.12}$$

Using eq. (A.1) and eq. (A.3) from the appendix, the dominant large G term is

$$\frac{1}{2}\int\frac{d^4Q}{(2\pi)^4}\frac{2\Pi_{\rm T}+\Pi_{\rm L}}{Q^2+\Pi_{\rm vac}} \simeq \frac{7\pi^2T^4g_{\rm eff}^2}{45}\int\frac{d^4Q}{(2\pi)^4}\frac{3(q^0)^2-q^2}{(Q^2)^3}\frac{1}{1-g_{\rm eff}^2\ln[(Q^2)/\overline{\mu}^2]/12\pi^2}\,. \tag{3.13}$$

This term is potentially logarithmically divergent. If one performs an angular average over the (euclidean) direction of Q, at fixed $|Q^2|$, then $(3(q^0)^2 - q^2)$ averages to zero, while the rest is a function of Q^2 only; so any regularization or cutoff procedure which respects euclidean invariance will be UV well behaved. However, if we perform the q^0 integration first, at fixed q, and then perform the d^3q integration, then because of the logarithm in the denominator, the result does not vanish; instead the q^0 integral gives a result of order $(g_{\rm eff}^2)^2T^4/q^3$. Therefore, performing the q^0 (or Minkowski ω) integration first and the d^3q integration last holds out the potential for fake logarithmic divergences.

Any sensible way to UV complete the theory, to deal with the Landau pole, will respect euclidean invariance. Therefore, however we deal with very large Q^2 values, we should do so in a euclidean invariant way. We choose to apply a large momentum cutoff procedure;[5] we stop the d^4Q integration at $Q^2 = a\Lambda^2_{\rm Landau}$, for $a<1$, varying the value of a between $1/4$ and $1/2$ to estimate the irreducible ambiguity. Applying a euclidean invariant high dimension operator cutoff works similarly.

As we just indicated, it is necessary to cut off the integration at a finite value of Q^2, lower than $\Lambda^2_{\rm Landau}$. This also requires some extra care with our Minkowski continuation;

[4]We should perform pieces linear in $n_{\rm b}$ in Minkowski space; but terms without $n_{\rm b}$ can be performed either in Minkowski space, or in euclidean space — integrating, not summing, on q^0 — or using some mixture, as convenient.

[5]The reader may worry that this will introduce gauge fixing dependence. It does not, within the class of Lorentz and Coulomb gauges, presumably because the self-energy is gauge invariant; recall that the nonabelian nature of the theory is not relevant at the order of interest.

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we must compute the great arcs at complex q^0 which connect the Minkowski and euclidean contours. Our procedure is this. First, we perform the Minkowski integration up to $\omega < \Lambda_{\rm Landau}\sqrt{a}$ and $q < q_{\rm max} \sim 25T$ (the contribution linear in $n_{\rm b}$ is carried to larger q). The Minkowski integration is done by numerical quadratures, using adaptive mesh refinement. Great care must be taken close to the light-cone at large q. To speed things up, the integrals required for the self-energies are expressed in terms of a handful of integrals of one variable (either $[\omega - q]$ or $[\omega + q]$), which are tabulated and spline interpolated so only two integrations nest. Then we add the great arc from the Minkowski to the euclidean frequency axis, and integrate over euclidean q^0 down to $(q^0)^2 + q^2 = q^2_{\rm max}$. This accounts for all $Q^2 < q^2_{\rm max}$. Then we integrate over euclidean $q^2_{\rm max} < Q^2 < a\Lambda^2_{\rm Landau}$, performing first the angular integration and then the integration over the magnitude of Q^2. Note that $(2\Pi_{\rm T}+\Pi_{\rm L})$ in eq. (3.12) vanishes on angular averaging at least to $O(T^{10})$. The lowest order term at large Q^2, after angular averaging, comes from $(2\Pi^2_{\rm T}+\Pi^2_{\rm L})^2/(Q^2+\Pi_{\rm vac})^2 \sim T^8$, so the irreducible Landau pole introduced ambiguity is $O(T^8/\Lambda^4_{\rm Landau})$, as claimed earlier.

We compute the pressure at a series of values of $g^2_{\rm eff}$, always at renormalization point

$$\overline{\mu}_{\rm DR} \equiv \pi e^{-\gamma_E} T \simeq 1.76T\,, \tag{3.14}$$

with $\gamma_E = 0.577215\ldots$ Euler's constant. This is the value suggested by dimensional reduction [24]; at this renormalization point, the energy density in an infrared magnetic field equals its tree level value, $\int d^3x\, B^2/2$. The result for $P_{\rm NLO}$ is T^4 times a pure function of $g^2_{\rm eff}(\overline{\mu}_{\rm DR})$,

$$P_{\rm NLO} = 2d_{\rm A}\frac{\pi^2}{90}T^4\,\mathcal{P}(g^2_{\rm eff}(\overline{\mu}_{\rm DR}))\,. \tag{3.15}$$

We have pulled out the free theory value; $2d_{\rm A} = 2(N_{\rm c}^2-1)$ counts the number of degrees of freedom in the gauge field, and $\pi^2 T^4/90$ is the usual result for a massless, bosonic field.

One may also compute the entropy density, which is the temperature derivative of the pressure;

$$S_{\rm NLO} = \frac{dP_{\rm NLO}}{dT} = 2d_{\rm A}\frac{2\pi^2}{45}T^3\left(\mathcal{P} + \frac{(g^2_{\rm eff})^2}{24\pi^2}\,\frac{d\mathcal{P}}{d\,g^2_{\rm eff}}\right). \tag{3.16}$$

This can be done from tabulated results for $\mathcal{P}$ by spline interpolation, for instance.

4. Results and discussion

We present our results for the pressure and entropy, plotted against $g^2_{\rm eff}$, in figure 3. The figure also shows pressure plotted against $\Lambda_{\rm Landau}/T$. As the coupling is increased, $\mathcal{P}$ at first falls, as expected (for instance in a quasiparticle picture); then it flattens out and rises, so at large $g^2_{\rm eff}$ the pressure exceeds the free theory value. The figure shows that the minimum of the pressure occurs when $\Lambda_{\rm Landau} > 10^4 T$; the point where it reaches the free theory value is $\Lambda_{\rm Landau} \simeq 110T$. Both occur at scales where the Landau singularity is still far off in the ultraviolet; they are robust predictions of the theory, regardless of the UV completion.

These results are firmly at odds with the ideal quasiparticle picture of the plasma presented by Peshier, Kämpfer, Pavlenko, and Soff [20], in which the pressure and entropy are described by treating the plasma as a gas of free but massive quasiparticles. In this

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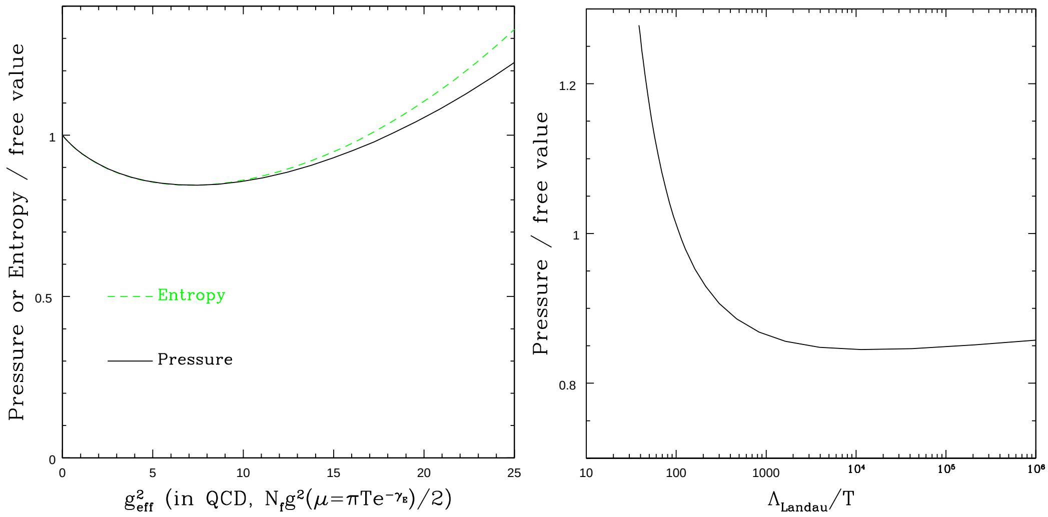

Figure 3: $P_{\rm NLO}$ (scaled by free value) plotted against $g^2_{\rm eff}$, left, and $\Lambda_{\rm Landau}/T$, right. The left plot also shows the entropy density. The right plot shows that the features indicating the breakdown of perturbation theory — the pressure bottoming out and becoming larger than the free theory value — occur long before the temperature reaches $\Lambda_{\rm Landau}$.

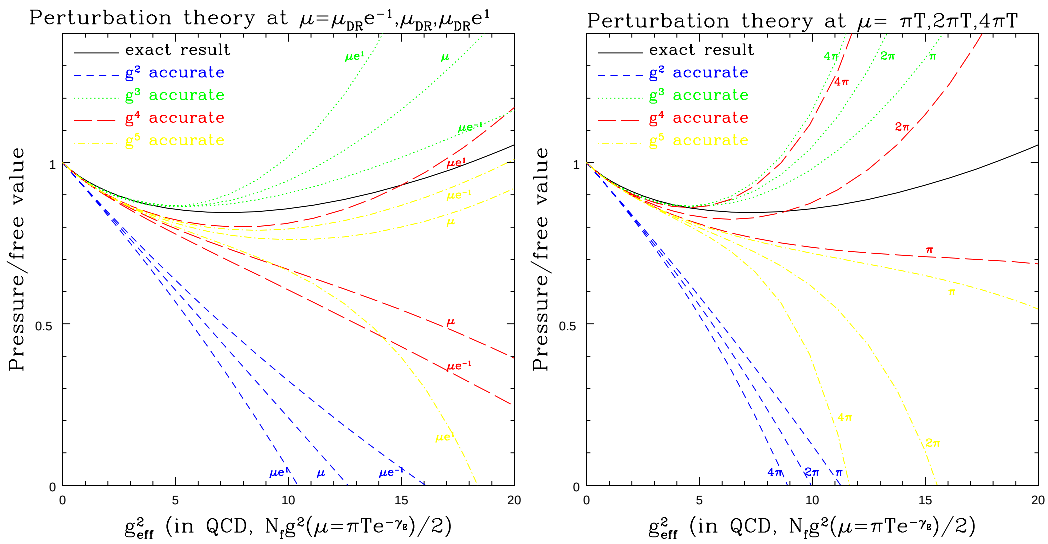

Figure 4: Exact result for $P_{\rm NLO}$, scaled by free value, presented in comparison to the results of perturbation theory. Each figure shows three choices for the renormalization point in perturbation theory; at left, they are within a power of e of $\overline{\mu}_{\rm DR}$; at right, they are within a factor of 2 of $2\pi T$.

model it is unexpected for $S_{\rm NLO}/T^3$ to rise as a function of $g^2_{\rm eff}$, because this requires that the quasiparticle masses get smaller with increasing coupling; and it is impossible to have $S_{\rm NLO}$ exceed its free theory value; a massive (quasi)particle contributes an entropy which is strictly less than a free massless particle contributes. Therefore we do not think that an ideal quasiparticle picture can be a good description of the physics of large $N_{\rm f}$ QCD. Since, as we have argued, large $N_{\rm f}$ is a subset of true QCD, we also don't expect it to be a good description of QCD.

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We also present the predictions of perturbation theory for the pressure, plotted against the actual value. The perturbative prediction for $P_{\rm LO}$ agrees with eq. (3.3). The NLO prediction, extracted from the results of Zhai and Kastening [6], is

$$P_{\rm NLO} = 2d_{\rm A}\,\frac{\pi^2 T^4}{90}\Bigg\{1 - \frac{25}{2}\,\frac{g^2_{\rm eff}}{16\pi^2} + \frac{80}{\sqrt{3}}\left(\frac{g^2_{\rm eff}}{16\pi^2}\right)^{3/2} + \left[\frac{100}{3}\ln\frac{\mu}{\overline{\mu}_{\rm DR}} - 68.907\right]\left(\frac{g^2_{\rm eff}}{16\pi^2}\right)^2 + \\ + \left[\frac{-320}{\sqrt{3}}\ln\frac{\mu}{\overline{\mu}_{\rm DR}} + \frac{160}{\sqrt{3}}\right]\left(\frac{g^2_{\rm eff}}{16\pi^2}\right)^{5/2} + O\left((g^2_{\rm eff})^3\right)\Bigg\}\,. \tag{4.1}$$

Note that, unlike in QCD, there is in principle no obstacle to computing the perturbative expansion to any order in $g^2_{\rm eff}$. Also there are no appearances of $\ln(g^2_{\rm eff})$, because the 3-D dimensionally reduced effective theory is free at order $N_{\rm f}^0$. We have evaluated each partial sum from this series, at a number of renormalization points (running the coupling from $\overline{\mu}_{\rm DR}$ using the exact relation, eq. (2.3)), in order to compare the perturbative expansion (and its renormalization point sensitivity) to the exact result. The comparison is shown in figure 4. As for QCD, the perturbative expansion is valid at weak coupling but becomes ill behaved long before the temperature reaches the Landau pole (or, in QCD, the phase transition temperature).

It is beyond the scope of the current paper to analyze large $N_{\rm f}$ QCD within each of the proposed resummation schemes, to compare to the exact results presented here. We leave this task to the authors of those schemes. However, to facilitate the comparison, we present our numerical results for the pressure at a number of values of $g^2_{\rm eff}$, in table 1.

What are the implications of our results for full QCD? Obviously it does not make sense to use large $N_{\rm f}$ QCD directly to model the behavior of real world QCD; after all, in the physically interesting range of temperatures, real world QCD has 3 flavors, which is definitely not $\gg N_{\rm c} = 3$. This was anyway not the point of studying large $N_{\rm f}$ QCD. Our object was to present a testing ground for other techniques. We expect these techniques to be accurate at small $g^2_{\rm eff}$ but, possibly, to break down at some larger value. What does that imply for their performance in real QCD? As we have tried to emphasize, success at large $N_{\rm f}$ probably does not ensure success in full QCD; but failure at large $N_{\rm f}$ almost certainly means the technique is not valid. But it would also be nice to have a rough way of equating a value of $g^2_{\rm eff}$ to a comparable value of the QCD coupling $\alpha_{\rm s}$, so if a technique works at weak coupling but later breaks down, we can estimate its range of validity in full QCD.

Probably the most reasonable way to equate $g^2_{\rm eff}$ with $\alpha_{\rm s}$ is to compare values of the Debye mass. At least, this makes sense in the picture where screening is the dominant physical process changing the pressure. The Debye mass for each theory, at leading order, is[6]

$$m_{\rm D}{}^2(N_{\rm f}\gg 1) \simeq \frac{g^2_{\rm eff}}{3}T^2\,, \qquad m_{\rm D}{}^2({\rm QCD}) \simeq \frac{2N_{\rm c}+N_{\rm f}}{6}g_{\rm s}^2T^2 = \frac{2\pi(2N_{\rm c}+N_{\rm f})}{3}\alpha_{\rm s}T^2\,. \tag{4.2}$$

Equating these, for 3 flavor QCD, gives $g^2_{\rm eff} = 18\pi\alpha_{\rm s}$. So $\alpha_{\rm s} = 0.11$ (relevant for temperatures of order 100GeV) is comparable to $g^2_{\rm eff} = 6$, whereas $\alpha_{\rm s} = .3$ (relevant closer to the QCD phase transition temperature) is comparable to $g^2_{\rm eff} = 17$.

[6]Had we chosen $\overline{\mu} = \overline{\mu}_{\rm DR}e^{1/2}$, the large $N_{\rm f}$ expression would be exact.

$g_{\rm eff}^2$	$\mathcal{P} = P_{\rm NLO}/P_{\rm NLO}(\rm free)$	$\Lambda_{\rm Landau}/T$
0.0	1.00000	∞
1.0	0.94621	2.1×10^{26}
2.0	0.91054	2.9×10^{13}
3.0	0.88508	1.5×10^{9}
4.0	0.8671	1.09×10^{7}
5.0	0.8551	$565,000$
6.0	0.8480	$78,000$
7.0	0.8451	$19,000$
8.0	0.8460	6650
9.0	0.8501	2920
10.0	0.8573	1510
11.0	0.8672	884
12.0	0.8796	564
13.0	0.8943	386
14.0	0.9113	279
15.0	0.9304	210
16.0	0.951	164
17.0	0.975	132
18.0	1.000	109
19.0	1.026	92
20.0	1.055	78
21.0	1.086	68
22.0	1.118	60
23.0	1.152	53
24.0	1.189	48
25.0	1.226	43

Table 1: NLO pressure, normalized to the free value, as a function of $g_{\rm eff}^2$. The numerical error is less than 2 in the last place shown, and is dominated by numerical issues; Landau pole ambiguity is at most 1 in the last digit in all entries.

In conclusion, we have presented exact results at next-to-leading order (first nontrivial order) for the pressure of large $N_{\rm f}$ QCD. These results should be useful for testing the reliability of resummed perturbative techniques, used for full QCD.

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A. Self-energies

The self-energies $\Pi_{\rm T}$, $\Pi_{\rm L}$ have been evaluated by Weldon [23]; here we present his results, rotated into euclidean space, and present a few terms of the expansion about small T^2/Q^2.

Following Weldon, we define two scalar functions, with normalization chosen to agree with his usage,

$$\begin{aligned} 2g_{\rm eff}^2\; G(Q) &\equiv \Pi^\mu_\mu(Q) - [{\rm vac}] = 2\Pi_{\rm T}(Q) + \Pi_{\rm L}(Q)\,, \\ 2g_{\rm eff}^2\; H(Q) &\equiv \Pi^{00}(Q) - [{\rm vac}] = \frac{q^2}{Q^2}\Pi_{\rm L}(Q)\,, \end{aligned} \tag{A.1}$$

so that

$$\begin{aligned} \Pi_{\rm L}(Q) &= 2g_{\rm eff}^2\,\frac{Q^2}{q^2}H(Q)\,, \\ \Pi_{\rm T}(Q) &= g_{\rm eff}^2\left(-\frac{Q^2}{q^2}H(Q) + G(Q)\right). \end{aligned} \tag{A.2}$$

Weldon's results for these scalar quantities, rotated into euclidean space and for general Q, are

$$\begin{aligned} G(Q) &= \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{e^{k/T}+1}\left[4k + \frac{Q^2}{2q}\ln\frac{(q^0)^2+(2k-q)^2}{(q^0)^2+(2k+q)^2}\right], \\ H(Q) &= \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{e^{k/T}+1}\left[2k + \frac{Q^2-4k^2}{4q}\ln\frac{(q^0)^2+(2k-q)^2}{(q^0)^2+(2k+q)^2} + \frac{q^0 k}{q}\, i\,\ln\frac{1+4k^2/(q^0-iq)^2}{1+4k^2/(q^0+iq)^2}\right]. \end{aligned}$$

Both expressions are pure real, despite the appearance of i in the expression for $H(Q)$. To obtain the retarded, Minkowski value, replace $q^0 \to i\omega{+}0$.

It is possible to expand each quantity in $T^2 \ll Q^2$ by series expanding the bracketed quantity in small k and then performing the k integration. This is expected to give an asymptotic series, since the k integration goes up to ∞ but with exponentially decaying weight. The next to leading order result, in euclidean space, is

$$\begin{aligned} G(Q\gg T) &\to \frac{7\pi^2(3(q^0)^2-q^2)}{45(Q^2)^2}T^4 - \frac{248\pi^4(5(q^0)^4 - 10(q^0)^2q^2 + q^4)}{315(Q^2)^4}T^6 + O(T^8)\,, \\ H(Q\gg T) &\to \frac{7\pi^2 q^2}{45(Q^2)^2}T^4 - \frac{248\pi^4 q^2(5(q^0)^2-q^2)}{945(Q^2)^4}T^6 + O(T^8)\,. \end{aligned} \tag{A.3}$$

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